AN EXAMPLE OF AN Sl_2-HILBERT SCHEME WITH MULTIPLICITIES

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ABSTRACT. We determine the invariant Hilbert scheme of the zero fibre of the moment map of an action of Sl_2 on $(\mathbb{C}^2)^{\oplus 6}$ as one of the first examples of invariant Hilbert schemes with multiplicities. While doing this, we present a general procedure how to realise the calculation of invariant Hilbert schemes, which have been introduced by Alexeev and Brion in [AB05]. We also consider questions of smoothness and connectedness and thereby show that our Hilbert scheme gives a resolution of singularities of the symplectic reduction of the action.

1. Introduction

Let G be a complex connected reductive algebraic group and X an affine G-scheme over $\mathbb{C}$. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible representations of G and let $h\colon \mathrm{Irr}(G) \to \mathbb{N}_0$ be a map, called *Hilbert function* in the following. In this setting, Alexeev and Brion define in [AB05] the invariant Hilbert scheme $\mathrm{Hilb}_h^G(X)$ parameterising G-invariant subschemes of X whose modules of global sections all have the same isotypic decomposition $\bigoplus_{\rho\in\mathrm{Irr}(G)} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V(\rho)$ as G-modules, where $V(\rho)$ denotes the G-module corresponding to the irreducible representation ρ. This generalises the G-Hilbert scheme of Ito and Nakamura [IN96].

In the case where the Hilbert function h is multiplicity-free, i.e. $h(\rho) \in \{0, 1\}$ for all $\rho \in \mathrm{Irr}(G)$, several examples of the invariant Hilbert scheme have been determined by Jansou [Jan07], Bravi and Cupit-Foutou [BCF08] and Papadakis and van Steirteghem [PvS10], which all turn out to be affine spaces. Jansou and Ressayre [JR09] give some examples of invariant Hilbert schemes with multiplicities, which are also affine spaces. There are some more involved examples of invariant Hilbert schemes by Brion (unpublished) and Budmiger [Bud10]. In this paper, we present a more complex example, namely of an Sl_2-Hilbert scheme with Hilbert function

$$h\colon \mathbb{N}_0 \to \mathbb{N},\ d \mapsto d+1. \tag{1}$$

The knowledge of such examples where the Hilbert scheme is not an affine space is important for understanding general properties of invariant Hilbert schemes: Which conditions have to be fulfilled so that the invariant Hilbert scheme is connected or smooth? Is the invariant Hilbert scheme a resolution of singularities of the quotient $X/\!\!/G$ as the G-Hilbert scheme is for finite G up to dimension 3 [BKR01]?

Our example of an Sl_2-Hilbert scheme will be smooth and connected and it even will be a resolution of singularities, but it does not inherit the additional structure of symplectic variety of the quotient.

Now let us present the setting of our example. Consider the action of Sl_2 on $(\mathbb{C}^2)^{\oplus 6} = \mathrm{Mat}_{2\times 6}(\mathbb{C})$ arising as symplectic double from the action of Sl_2 on $(\mathbb{C}^2)^{\oplus 3}$ via multiplication on the left.

Date: October 18, 2010.

This work has been partially supported by DAAD and by the SFB/TR 45 "Periods, Moduli Spaces and Arithmetic of Algebraic Varieties" of the DFG.

Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$. The moment map $\mu\colon (\mathbb{C}^2)^{\oplus 6} \to \mathfrak{sl}_2$, $M \mapsto MQM^tJ$ defines the symplectic reduction $(\mathbb{C}^2)^{\oplus 6}/\!/\!/Sl_2 := \mu^{-1}(0)/\!/Sl_2$. In [Bec10] we obtained its description as a nilpotent orbit closure $\mu^{-1}(0)/\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]}$ in $\mathfrak{so}_6$. Writing $(\mathbb{C}^2)^{\oplus 6} = \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^6$ we see that we have a symmetric situation with an action of $SO_6 = SO(Q)$ by multiplication from the right and μ is invariant for this action, so that SO_6 acts on the zero fibre $\mu^{-1}(0)$ and as both actions commute, SO_6 also acts on the quotient by Sl_2. The quotient map $\nu\colon \mu^{-1}(0) \to \mu^{-1}(0)/\!/Sl_2$ is given by mapping M to M^tJMQ. In fact, the quotient map of the Sl_2-action is the moment map of the SO_6-action and vice versa. The SO_6-action will play an important role while analysing $\mu^{-1}(0)/\!/Sl_2$ and the corresponding Hilbert scheme.

There are two well-known symplectic resolutions of singularities of the symplectic variety $\overline{\mathcal{O}}_{[2^2,1^2]}$, namely the cotangent bundle $T^*\mathbb{P}^3 \cong \{(A, L) \in Y \times \mathbb{P}^3 \mid \operatorname{im} A \subset L\}$ and its dual $(T^*\mathbb{P}^3)^* \cong \{(A, H) \in Y \times (\mathbb{P}^3)^* \mid H \subset \ker A\}$, where $Y = \{A \in \mathfrak{sl}_4 \mid \operatorname{rk} B \leq 1\} \cong \overline{\mathcal{O}}_{[2^2,1^2]}$. We want to know if there is a distinguished (symplectic) resolution. Since Hilbert schemes of points and G-Hilbert schemes are often candidates for (symplectic) resolutions [Fog68, Bea83, BKR01], we hope that this is also true for invariant Hilbert schemes. Indeed, with the choice of the Hilbert function (1), in our example we find

Theorem 1.1. *The invariant Hilbert scheme* $Sl_2\text{-Hilb}(\mu^{-1}(0)) := \operatorname{Hilb}_h^{Sl_2}(\mu^{-1}(0))$ *of the zero fibre of the moment map of the action of* Sl_2 *on* $(\mathbb{C}^2)^{\oplus 6}$ *is the scheme*

$$\{(A, W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \operatorname{Grass}_{iso}(2, \mathbb{C}^6) \mid \operatorname{im} A^t \subset W\}. \tag{2}$$

It is smooth and connected, and thus a resolution of singularities of the symplectic reduction $\mu^{-1}(0)/\!/Sl_2$.

Remark. $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is itself not a *symplectic* resolution of $\mu^{-1}(0)/\!/Sl_2$, but as it is isomorphic to $\{(A, L, H) \in Y \times \mathbb{P}^3 \times (\mathbb{P}^3)^* \mid \operatorname{im} A \subset L \subset H \subset \ker A\}$ via $L = \{v \in \mathbb{C}^4 \mid \dim(v \wedge W) = 0\}$, $H = \{v \in \mathbb{C}^4 \mid \dim(v \wedge W) \leq 1\}$ and $W = L \wedge H$, it dominates the two symplectic resolutions:

$$\begin{tikzcd}
 & Sl_2\text{-Hilb}(\mu^{-1}(0)) \arrow[dl] \arrow[dd] \arrow[dr] & \\
T^*\mathbb{P}^3 \arrow[dr] & & (T^*\mathbb{P}^3)^* \arrow[dl] \\
 & \mu^{-1}(0)/\!/Sl_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (A, W) \arrow[dl, mapsto] \arrow[dd, mapsto] \arrow[dr, mapsto] & \\
(A, L) \arrow[dr, mapsto] & & (A, H) \arrow[dl, mapsto] \\
 & A &
\end{tikzcd}$$

This paper is organised as follows: In the second chapter we introduce the invariant Hilbert scheme as defined by Alexeev and Brion in [AB05], building upon the work of Haiman and Sturmfels on the multigraded Hilbert scheme [HS04]. We give their definition of the invariant Hilbert functor, which is represented by the invariant Hilbert scheme, we introduce the Hilbert-Chow morphism and analyse which conditions on the Hilbert function have to be satisfied so that this morphism, or at least its restriction to a certain component, is proper and birational, the important properties for being a resolution. With regard to this, we define the orbit component $\operatorname{Hilb}_h^G(X)^{orb}$, which is the unique component mapping birationally to the set of closed G-orbits. If the invariant Hilbert scheme is not irreducible, this component is still a candidate for a resolution.
Afterwards, we turn to our example in chapter 3. First, we compute the general fibre of the quotient in order to determine the right Hilbert function which guarantees birationality.

The forth chapter is the heart of this article. First we show how to find generators of the locally free sheaves occurring in the definition of the invariant Hilbert functor in general, then we construct an embedding of the Hilbert scheme into a product of Grassmannians by ideas of Brion and based on the embedding constructed in [HS04]. Thus this note not only gives a complex example of an invariant Hilbert scheme with multiplicities of a variety which is not an affine space, but it also can be consulted as a guidance for the determination of further examples. While describing the general process we always switch to its application to the example at the end of each step. As a result of this, we obtain the orbit component in our example as (2).
To conclude the proof of theorem 1.1, i.e. to find out if the orbit component coincides with the whole Hilbert scheme, in chapter 5 we show that the latter is smooth by considering the tangent space to the invariant Hilbert scheme and we prove that it is connected.

Acknowledgements. I am very thankful to Michel Brion for introducing me to the world of invariant Hilbert schemes and for guiding me through the determination of this example. I also would like to thank him for his hospitality during the four months I spent in Grenoble. I thank Manfred Lehn and Christoph Sorger for proposing me the work on G- and invariant Hilbert schemes and for several discussions about the example. I thank Ronan Terpereau for the exchange of knowledge on invariant Hilbert schemes. I am greatful to José Bertin for his private lessons on the G-Hilbert scheme which also enlarged my understanding of invariant Hilbert schemes. I greatfully acknowledge the financial support by DAAD and SFB/TR 45.

2. The invariant Hilbert scheme after Alexeev and Brion

Before passing to the specific example of an invariant Hilbert scheme, we present in general the construction of the invariant Hilbert scheme introduced by Alexeev and Brion in [AB04, AB05], which generalizes the G-Hilbert scheme for finite groups G after Ito and Nakamura [IN96, IN99]. For further details on invariant Hilbert schemes consult Brion's survey [Bri10].

Let G be a complex reductive algebraic group and X an affine G-scheme over $\mathbb{C}$. Let $\mathrm{Irr}(G)$ denote the set of isomorphism classes of irreducible representations of G and denote by $\rho_0 \in \mathrm{Irr}(G)$ the trivial representation. As G is reductive every G-module W decomposes as a sum of its isotypic components $W = \bigoplus_{\rho\in\mathrm{Irr}\,G} W_{(\rho)} = \bigoplus_{\rho\in\mathrm{Irr}\,G} W_\rho \otimes_{\mathbb{C}} V(\rho)$, where $W_\rho = \mathrm{Hom}_G(V(\rho), W)$.
We call the dimension of $\mathrm{Hom}_G(V(\rho), W)$ the *multiplicity* of ρ in W. If each irreducible representation occurs with finite multiplicity, i.e. for all $\rho \in \mathrm{Irr}\,G$ we have $h(\rho) := \dim \mathrm{Hom}_G(V(\rho), W) < \infty$, then $h\colon \mathrm{Irr}(G) \to \mathbb{N}$ is called the *Hilbert function* of W.
If $\mathcal{F}$ is a coherent G-sheaf over some noetherian basis S where G acts trivially, there is also an isotypic decomposition $\mathcal{F} = \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V(\rho)$, where the sheaves of covariants $\mathcal{F}_\rho = \mathcal{H}om^G(V(\rho), \mathcal{F})$ are coherent $\mathcal{O}_S$-modules. They are locally free of rank $h(\rho)$ if and only if $\mathcal{F}$ is flat over S.

Definition 2.1. [AB05, Def. 1.5] For any function $h\colon \mathrm{Irr}\,G \to \mathbb{N}_0$, the associated functor

$$\mathcal{H}ilb_h^G(X)\colon (\mathrm{Schemes})^{\mathrm{op}} \to (\mathrm{Sets})$$

$$S \mapsto \left\{ \begin{array}{c|l} \begin{tikzcd} Z \subset X \times S \arrow[d, "pr_2"] \arrow[dr, "p"'] & \\ S & \end{tikzcd} & \begin{array}{l} Z \text{ a } G\text{–invariant closed subscheme,} \\ p \text{ flat,} \\ p_*\mathcal{O}_Z \cong \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V(\rho) \end{array} \end{array} \right\},$$

$$(f\colon T \to S) \mapsto (Z \mapsto (id \times f)^* Z)$$

such that the sheaves of covariants $\mathcal{F}_\rho = \mathcal{H}om^G(V(\rho), p_*\mathcal{O}_Z)$ are locally free $\mathcal{O}_S$–modules of rank $h(\rho)$, is called the *invariant Hilbert functor*.

Remark. In analogy to the case of finite G the coordinate ring of every fibre Z_s of the projection $p\colon Z \to S$ of a closed point $s \in S$ satisfies

$$\mathbb{C}[Z_s] = \Gamma(Z_s, \mathcal{O}_{Z_s}) = (p_*\mathcal{O}_Z)(s) \cong \bigoplus_{\rho \in \operatorname{Irr} G} \mathbb{C}^{h(\rho)} \otimes_{\mathbb{C}} V(\rho)$$

since the fibre $\mathcal{F}_\rho(s)$ is a $\mathbb{C}$-vector space of dimension $h(\rho)$. This can be considered as $h(\rho)$ copies of $V(\rho)$ for every $\rho \in \operatorname{Irr} G$, so we write $\bigoplus_{\rho \in \operatorname{Irr} G} h(\rho)V(\rho)$ instead. In particular, the only invariants of $\mathbb{C}[Z_s]$ are the elements of the isotypical component of the trivial representation ρ_0, i.e. $h(\rho_0)$ copies of the constants.

Proposition 2.2. [HS04, AB04, AB05] *There exists a quasi-projective scheme* $\operatorname{Hilb}^G_h(X)$ *representing* $\mathcal{H}ilb^G_h(X)$*, the* invariant Hilbert scheme.

There is an analogue of the Hilbert-Chow morphism, the quotient-scheme map

$$\eta\colon \operatorname{Hilb}^G_h(X) \to \operatorname{Hilb}^{h(\rho_0)}(X/\!\!/G),\ Z \mapsto Z/\!\!/G,$$

described in [Bri10, § 3.4]. It is proper and even projective [Bri10, Prop. 3.12]. If we add the condition $h(\rho_0) = 1$, then we have $\eta\colon \operatorname{Hilb}^G_h(X) \to \operatorname{Hilb}^1(X/\!\!/G) = X/\!\!/G$. We will always assume this in the following. For birationality one has to choose the Hilbert function $h = h_X$ defined by the isotypic decomposition of the general fibre F of the quotient map $\nu\colon X \to X/\!\!/G$:

$$\Gamma(F, \mathcal{O}_F) = \bigoplus_{\rho \in \operatorname{Irr} G} h_X(\rho)V(\rho).$$

Lemma 2.3. *If X is irreducible, there is an irreducible component* $\operatorname{Hilb}^G_{h_X}(X)^{orb}$ *of* $\operatorname{Hilb}^G_{h_X}(X)$ *such that the restriction of the Hilbert-Chow morphism* $\eta\colon \operatorname{Hilb}^G_{h_X}(X)^{orb} \to X/\!\!/G$ *is birational.*

Proof By an independent result of Brion [Bri10, Prop. 3.15] and Budmiger [Bud10, Thm I.1.1], if $\nu\colon X \to X/\!\!/G$ is flat, then $X/\!\!/G$ represents the Hilbert functor $\mathcal{H}ilb^G_{h_X}(X)$, thus $X/\!\!/G \cong \operatorname{Hilb}^G_{h_X}(X)$. In the non-flat case let $U \subset X/\!\!/G$ be a non-empty open affine subset such that $\nu^{-1}(U) \to U$ is flat. Then $h_{\nu^{-1}(U)} = h_X$ since all fibres of $\nu^{-1}(U) \to U$ have the same Hilbert function as the general fibre of ν, so U is isomorphic to the open subscheme $\operatorname{Hilb}^G_{h_X}(\nu^{-1}(U)) = \eta^{-1}(U)$ of $\operatorname{Hilb}^G_{h_X}(X)$. Thus the restriction of η to its closure $\operatorname{Hilb}^G_{h_X}(X)^{orb} := \overline{\eta^{-1}(U)}$ is birational.
If X and hence $X/\!\!/G$ is irreducible, so is U and $\eta^{-1}(U) \cong U$. Hence there is an irreducible component $C \subset \operatorname{Hilb}^G_{h_X}(X)$ containing $\eta^{-1}(U)$. The morphism $\eta|_C\colon C \to X/\!\!/G$ is dominant and the fibres of an open subset of $X/\!\!/G$ are finite (indeed the preimage of each element in U is a point). This means that $\dim C = \dim X/\!\!/G$, hence $\overline{\eta^{-1}(U)} = C$ is an irreducible component. □

Definition 2.4. The variety $\operatorname{Hilb}^G_{h_X}(X)^{orb}$ constructed in the lemma is called the *orbit component* or *main component* of $\operatorname{Hilb}^G_{h_X}(X)$. It corresponds to the coherent component for toric Hilbert schemes and is the principal component in the sense that it is birational to the quotient $X/\!\!/G$ parameterising the closed orbits of the action of G on X.

Remark. The map $\eta|_{\operatorname{Hilb}^G_{h_X}(X)^{orb}}$ is dominant and proper and $\operatorname{Hilb}^G_{h_X}(X)^{orb} \subset \operatorname{Hilb}^G_{h_X}(X)$ is closed, so $\eta|_{\operatorname{Hilb}^G_{h_X}(X)^{orb}}$ is even surjective.

Remark 2.5. If the general fibre of $\nu\colon X \to X/\!\!/G$ happens to be the group G itself, the Hilbert function is $h_X(\rho) = \dim(V(\rho))$ since we have $\Gamma(G, \mathcal{O}_G) = \mathbb{C}[G] = \bigoplus_{\rho \in \operatorname{Irr} G} V(\rho)^* \otimes_{\mathbb{C}} V(\rho)$ and $\dim(V(\rho)^*) = \dim(V(\rho))$. In analogy to the case of finite groups we write in this situation

$$G\text{-Hilb}(X) := \operatorname{Hilb}^G_{h_X}(X) \qquad \text{and} \qquad G\text{-Hilb}(X)^{orb} := \operatorname{Hilb}^G_{h_X}(X)^{orb}.$$

3. Determination of the Hilbert function

3.1. **The quotient related to the Hilbert scheme.** The action of Sl_2 on $(\mathbb{C}^2)^{\oplus 3}$ via multiplication on the left is self-dual, so its symplectic double $Sl_2 \times (\mathbb{C}^2)^{\oplus 6} \to (\mathbb{C}^2)^{\oplus 6}$ is also given by multiplication from the left $(g, M) \mapsto gM$. We would like the symplectic structure on $(\mathbb{C}^2)^{\oplus 6}$ to descend to the quotient, so instead of $(\mathbb{C}^2)^{\oplus 6}/\!\!/Sl_2$ we consider the symplectic reduction $(\mathbb{C}^2)^{\oplus 6}/\!\!/\!\!/Sl_2 = \mu^{-1}(0)/\!\!/Sl_2$, defined as the quotient of the zero fibre of the moment map $\mu\colon (\mathbb{C}^2)^{\oplus 6} \to \mathfrak{sl}_2$, $M \mapsto MQM^tJ$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 0 & I_3 \\ I_3 & 0 \end{smallmatrix}\right)$. For a more detailed discussion of this action we refer to [Bec10], as well as for the description as a nilpotent orbit closure $\mu^{-1}(0)/\!\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]} = \{A \in \mathfrak{so}_6 \mid A^2 = 0,\ \operatorname{rk} A \le 2,\ \operatorname{Pf}_4(QA) = 0\}$, where $\operatorname{Pf}_4(QA)$ denotes the Pfaffians of the 15 skew-symmetric 4×4-minors of QA. Under the adjoint action this variety consists of two orbits of matrices of rank 2 and 0, respectively: $\overline{\mathcal{O}}_{[2^2,1^2]} = \mathcal{O}_{[2^2,1^2]} \cup \{0\}$.
The quotient map is $\nu\colon \mu^{-1}(0) \to \overline{\mathcal{O}}_{[2^2,1^2]}$, $M \to M^tJMQ$.
In coordinates $M = \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \end{smallmatrix}\right)$ we have

$$\begin{aligned} M^tJMQ &= \begin{pmatrix} (-x_{2,i}x_{1,3+j} + x_{1,i}x_{2,3+j})_{ij} & (-x_{2,i}x_{1,j} + x_{1,i}x_{2,j})_{ij} \\ (-x_{2,3+i}x_{1,3+j} + x_{1,3+i}x_{2,3+j})_{ij} & (-x_{2,3+i}x_{1,j} + x_{1,3+i}x_{2,j})_{ij} \end{pmatrix} \\ &= \begin{pmatrix} (\Lambda^{i,3+j})_{ij} & (\Lambda^{i,j})_{ij} \\ (\Lambda^{3+i,3+j})_{ij} & (\Lambda^{j,3+i})_{ij} \end{pmatrix}, \end{aligned}$$

where i and j always range from 1 to 3 and $\Lambda^{s,t} = \det(x^{(s)}, x^{(t)})$ is the 2×2-minor of the s-th and t-th column in M. Thus the fibres of ν consist of those M with fixed 2×2-minors. A further condition is $M \in \mu^{-1}(0)$, i.e.

$$0 = MQM^t = \begin{pmatrix} 2 \cdot \sum\limits_{i=1}^{3} x_{1,i}x_{1,3+i} & \sum\limits_{i=1}^{3} (x_{1,i}x_{2,3+i} + x_{1,3+i}x_{2,i}) \\ \sum\limits_{i=1}^{3} (x_{1,i}x_{2,3+i} + x_{1,3+i}x_{2,i}) & 2 \cdot \sum\limits_{i=1}^{3} x_{2,i}x_{2,3+i} \end{pmatrix}.$$

3.2. **The general fibre of the quotient.** In order to determine the Hilbert function $h_{\mu^{-1}(0)}$, so that $Sl_2\text{-Hilb}(\mu^{-1}(0)) = \operatorname{Hilb}^{Sl_2}_{h_{\mu^{-1}(0)}}(\mu^{-1}(0))$ birational to the quotient $\mu^{-1}(0)/\!\!/Sl_2$, we have to compute the general fibre of ν. Therefore we need to know the locus where the quotient is flat.

Proposition 3.1. *The quotient ν restricted to the preimage of the open orbit of the SO_6-action $\nu^{-1}(\mathcal{O}_{[2^2,1^2]}) \to \mathcal{O}_{[2^2,1^2]}$ is flat and the fibres over all points in the orbit $\mathcal{O}_{[2^2,1^2]}$ are isomorphic.*

Proof $\mu^{-1}(0)$ is equipped with an action of SO_6 via multiplication on the right, which induces the adjoint action on $\mu^{-1}(0)/\!\!/Sl_2$. Since $\nu\colon \mu^{-1}(0) \to \mu^{-1}(0)/\!\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]}$ is SO_6-equivariant, ν is flat over the whole SO_6-orbit $\mathcal{O}_{[2^2,1^2]}$ or over no point of this orbit. By Grothendieck's lemma on generic flatness and $\overline{\mathcal{O}}_{[2^2,1^2]} \setminus \mathcal{O}_{[2^2,1^2]} = \{0\}$ the second case cannot occur. By equivariance, all fibres over this orbit are isomorphic. □

As a consequence, for computing the general fibre it is enough to determine the fibre over one point A_0 in the flat locus $\mathcal{O}_{[2^2,1^2]}$. We choose $A_0 = (a_{ij})$ with $a_{15} = -a_{24} = 1$ and $a_{ij} = 0$ otherwise.

For $M \in \nu^{-1}(A_0)$ this corresponds to $\Lambda^{1,2} = 1$, $\Lambda^{i,j} = 0$ otherwise. Thus

$$1 = \Lambda^{1,2} = x_{11}x_{22} - x_{12}x_{21}, \quad \text{hence } x_{11} \neq 0 \neq x_{22} \text{ or } x_{12} \neq 0 \neq x_{21}.$$

Without loss of generality assume $x_{11} \neq 0$. Then $\quad x_{22} = \dfrac{1 + x_{12}x_{21}}{x_{11}}$.
For $j = 3, \dots, 6$ we have

$$\begin{aligned}
0 = \Lambda^{1,j} = x_{11}x_{2j} - x_{1j}x_{21} \qquad &\Rightarrow \quad x_{2j} = \frac{x_{1j}x_{21}}{x_{11}}, \\
0 = \Lambda^{2,j} = x_{12}x_{2j} - x_{1j}x_{22} \qquad &\Rightarrow \quad x_{12}\frac{x_{1j}x_{21}}{x_{11}} = x_{1j}\frac{1 + x_{12}x_{21}}{x_{11}} = \frac{x_{1j}}{x_{11}} + \frac{x_{1j}x_{12}x_{21}}{x_{11}} \\
&\Rightarrow \quad x_{1j} = 0 \quad \text{for } j = 3, \dots, 6, \\
&\Rightarrow \quad x_{2j} = \frac{x_{1j}x_{21}}{x_{11}} = 0 \quad \text{for } j = 3, \dots, 6.
\end{aligned}$$

This implies
$$\begin{aligned}
&x_{11}x_{14} + x_{12}x_{15} + x_{13}x_{16} = 0, \\
&x_{11}x_{24} + x_{12}x_{25} + x_{13}x_{26} + x_{14}x_{21} + x_{15}x_{22} + x_{16}x_{23} = 0, \\
&x_{21}x_{24} + x_{22}x_{25} + x_{23}x_{26} = 0,
\end{aligned}$$

so $M \in \mu^{-1}(0)$ is automatic. This shows that the general fibre is

$$F := \nu^{-1}(A_0) = \left\{ \left(\begin{smallmatrix} x_{11} & x_{12} & 0 & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 & 0 & 0 \end{smallmatrix}\right) \in (\mathbb{C}^2)^{\oplus 6} \,\middle|\, x_{11}x_{22} - x_{12}x_{21} = 1 \right\} \cong Sl_2.$$

Remark. Analogous calculations over 0 show that the fibre $\nu^{-1}(0)$ has dimension 5, so ν is not flat over 0 and $\mathcal{O}_{[2^2,1^2]}$ is the maximal flat locus.

3.3. **The Hilbert function of the general fibre.** The Hilbert function is determined by the isotypic decomposition of the general fibre.
The irreducible representations of Sl_2 are parametrised by the natural numbers: $\operatorname{Irr}(Sl_2) \cong \mathbb{N}_0$, $V_d \leftrightarrow d$, where $V_d = \mathbb{C}[x,y]_d$ consists of homogeneous polynomials of degree d so that $\dim V_d = d + 1$. By remark 2.5 the coordinate ring of Sl_2 decomposes as

$$\mathbb{C}[Sl_2] = \bigoplus_{d \in \mathbb{N}_0} (\dim V_d) V_d = \bigoplus_{d \in \mathbb{N}_0} (d+1) V_d,$$

so in this case the Hilbert function is given by the dimension $h_{\mu^{-1}(0)}(d) = \dim V_d = d + 1$. For the Hilbert scheme this means that the sheaves $\mathcal{F}_d$ have to be locally free of rank $d + 1$.

4. Determination of the orbit component

Our idea to identify Sl_2-Hilb$(\mu^{-1}(0))$ is to determine generators for the sheaves of covariants $\mathcal{F}_d$ and to use them to embed the Sl_2-Hilbert scheme into the product of $\mu^{-1}(0)/\!\!/Sl_2$ and some Grassmannian. First, in section 4.1 we describe the sheaves $\mathcal{F}_\rho$ in general by giving a space of generators F_ρ as an $\mathcal{O}_{\mathrm{Hilb}_h^G(X)}$-module and we calculate $\mathcal{F}_1$ in our example. In section 4.2 we describe how to obtain a map η_ρ to the Grassmannian of quotients of F_ρ of rank $h(\rho)$ for each $\rho \in \operatorname{Irr} G$. We show that one can embed $\mathrm{Hilb}_h^G(X)$ into a product of finitely many of these Grassmannians. Afterwards, for Sl_2-Hilb$(\mu^{-1}(0))$ we calculate the map η_1 corresponding to the standard representation and we show that this single representation is enough to give an embedding of the orbit component into $\mu^{-1}(0)/\!\!/Sl_2 \times \mathrm{Grass}(F_1, h(1))$. Then we determine a strict subset of this which contains the image. Finally, by writing the Grassmannian as a homogeneous space we prove in section 4.3 that the embedding is even an isomorphism. Since the elements of Sl_2-Hilb$(\mu^{-1}(0))$ are subschemes of $\mu^{-1}(0)$, we explicitly determine these subschemes in section 4.4.

4.1. **The sheaves of covariants $\mathcal{F}_\rho$.** To describe the invariant Hilbert scheme or at least its orbit component, we have to determine all possibilities of locally free sheaves $\mathcal{F}_\rho$ of rank $h(\rho)$ on $\mathrm{Hilb}_h^G(X)$. For the trivial representation we have the following result by Brion [Bri10, Proof of Prop. 3.15], for which we give a more detailed proof.

Lemma 4.1. *If $h(\rho_0) = 1$ then for any scheme S and every subscheme $Z \in \mathcal{H}ilb_h^G(X)(S)$ we have $\mathcal{F}_{\rho_0} = \mathcal{O}_S$. In particular, for the universal subscheme $\mathcal{F}_{\rho_0} = \mathcal{O}_{\mathrm{Hilb}_h^G(X)}$.*

Proof Taking invariants, the defining equation of the $\mathcal{F}_\rho$ implies $p_*\mathcal{O}_Z^G = \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathcal{F}_\rho \otimes_{\mathbb{C}} V(\rho)^G$. But the trivial representation is the only irreducible representation admitting invariants, and all of its elements are invariants. Thus $\bigoplus_{\rho\in\mathrm{Irr}(G)} \mathcal{F}_\rho \otimes_{\mathbb{C}} V(\rho)^G = \mathcal{F}_{\rho_0}$ and there is a morphism $p^\# \colon \mathcal{O}_S = \mathcal{O}_S^G \to p_*\mathcal{O}_Z^G = \mathcal{F}_{\rho_0}$ induced by p, which is injective since p is surjective. Both sides are locally free $\mathcal{O}_S$-modules of rank one. Over each closed point $s \in S$ the fibres are $\mathcal{O}_S(s) = \mathcal{O}_{S,s}/\mathfrak{m}_s = k(s) = \mathbb{C}$ and $\mathcal{F}_{\rho_0}(s) = (p_*\mathcal{O}_Z)^G(s) = (p_*\mathcal{O}_Z)^G \otimes_{\mathbb{C}} k(s) = (p_*\mathcal{O}_Z \otimes_{\mathbb{C}} k(s))^G = \mathbb{C}[Z_s]^G$, and $\mathbb{C}[Z_s]^G = V(\rho_0) \cong \mathbb{C}$. So by Nakayama's lemma, $p^\#$ is an isomorphism, hence $\mathcal{O}_S \cong \mathcal{F}_{\rho_0}$. □

For general ρ, we additionally observe what happens if there is an action on X by another complex connected reductive group H commuting with the G-action. By [Bri10, Prop. 3.10], such an action also induces an action on $X/\!\!/G$ and on $\mathrm{Hilb}_h^G(X)$, such that the quotient map and the Hilbert-Chow morphism are H-equivariant.
Consider the isotypic decomposition $\mathbb{C}[X] = \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathbb{C}[X]_\rho \otimes_{\mathbb{C}} V(\rho)$, where H acts by the induced action on $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V(\rho), \mathbb{C}[X])$ and trivially on $V(\rho)$.

Proposition 4.2. *For every $\rho \in \mathrm{Irr}\,G$, the $\mathbb{C}[X]^G$-module $\mathbb{C}[X]_\rho$ is finitely generated, so there is a finite dimensional H-module F_ρ and an H-equivariant surjection $\mathbb{C}[X]^G \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathbb{C}[X]_\rho$. The space F_ρ generates $\mathcal{F}_\rho$ as an $\mathcal{O}_S$-module for every scheme S and gives a morphism of $\mathcal{O}_S$-H-modules $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathcal{F}_\rho$.*

Proof The space $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V(\rho), \mathbb{C}[X])$ is finitely generated as an $\mathbb{C}[X]^G$-module, see [Dol03, Cor. 5.1]. Thus we can choose finitely many generators and define F_ρ to be the H-module generated by them. This gives an H-equivariant surjection $\mathbb{C}[X]^G \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathbb{C}[X]_\rho$.
To determine generators for $\mathcal{F}_\rho$ we use the universal subscheme $\mathrm{Univ}_h^G(X)$. Then we obtain the result for an arbitrary scheme S and every element in $\mathcal{H}ilb_h^G(X)(S)$ by pulling it back. We have

$$\begin{tikzcd}
\mathrm{Univ}_h^G(X) \arrow[drr, "p"'] & \subset & X \times \mathrm{Hilb}_h^G(X) \arrow[r] \arrow[d, "pr_2"] & X \arrow[d, "\nu"] \\
& & \mathrm{Hilb}_h^G(X) \arrow[r, "\eta"] & X/\!\!/G
\end{tikzcd}$$

The action of H on X, $X/\!\!/G$ and $\mathrm{Hilb}_h^G(X)$ also induces an action of H on $X \times_{X/\!\!/G} \mathrm{Hilb}_h^G(X)$ and $\mathrm{Univ}_h^G(X)$ such that all morphisms in the diagram are H-equivariant. By [Bri10, Prop. 3.15], the diagram commutes and hence $\mathrm{Univ}_h^G(X)$ is even contained in $X \times_{X/\!\!/G} \mathrm{Hilb}_h^G(X)$. This inclusion yields a surjective H-equivariant morphism

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X] \twoheadrightarrow p_*\mathcal{O}_{\mathrm{Univ}_h^G(X)}.$$

By definition, we have $p_*\mathcal{O}_{\mathrm{Univ}_h^G(X)} = \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{F}_\rho \otimes_{\mathbb{C}} V(\rho)$ with an induced action of H on each $\mathcal{F}_\rho$ and the trivial action on $V(\rho)$. Furthermore, we can consider the isotypic decomposition $\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X] = \bigoplus_{\rho\in\mathrm{Irr}\,G} \mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \otimes_{\mathbb{C}} V(\rho)$ as G-modules. Together, we

obtain H-equivariant surjections

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \twoheadrightarrow \mathcal{F}_\rho$$

for every $\rho \in \mathrm{Irr}(G)$. This shows that the $\mathcal{O}_{\mathrm{Hilb}_h^G(X)}$-$H$–module $\mathcal{F}_\rho$ is generated by $\mathbb{C}[X]_\rho$, which is in turn generated by F_ρ over $\mathbb{C}[X]^G$. This yields

$$\mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}} F_\rho \twoheadrightarrow \mathcal{O}_{\mathrm{Hilb}_h^G(X)} \otimes_{\mathbb{C}[X]^G} \mathbb{C}[X]_\rho \twoheadrightarrow \mathcal{F}_\rho. \tag{3}$$

□

Application to $\mathcal{F}_1$. Now we apply this to our example. We know that V_0 is the trivial representation and by lemma 4.1 $\mathcal{F}_0 = \mathcal{O}_{Sl_2\text{-Hilb}(\mu^{-1}(0))}$ is free of rank 1. We suppose that for the representation of lowest dimension $\mathcal{F}_d$ is easiest to compute, so we begin with the standard representation $V_1 = \mathbb{C}^2$. It will turn out in proposition 4.4 that at least the orbit component $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ is already completely determined by this sheaf.
There is an action of SO_6 on $\mu^{-1}(0)$ via multiplication from the right and the induced action on $\overline{\mathcal{O}}_{[2^2,1^2]}$ by conjugation. The induced action on $Sl_2\text{-Hilb}(\mu^{-1}(0))$ is also by multiplication from the right. Following proposition 4.2 we obtain

Proposition 4.3. *$\mathcal{F}_1$ is generated by the six projections $p_i|_{\mu^{-1}(0)}\colon \mu^{-1}(0) \to \mathbb{C}^2$, $i = 1, \dots, 6$. Hence we may take $F_1 \cong \mathbb{C}^6$ the standard representation of SO_6.*

Proof Because of proposition 4.2 and self-duality of the standard representation of Sl_2, $\mathcal{F}_1$ is generated by $\mathrm{Hom}_{Sl_2}(\mathbb{C}^2, \mathbb{C}[\mu^{-1}(0)]) = \mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2)$. The inclusion $\mu^{-1}(0) \subset (\mathbb{C}^2)^{\oplus 6}$ induces a surjection $\mathrm{Mor}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2) \twoheadrightarrow \mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2)$ by shrinking morphisms to $\mu^{-1}(0)$. By [How95], the space of Sl_2-equivariant morphisms $\mathrm{Mor}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$ is a free module of rank 6 over the ring of invariants $\mathbb{C}[(\mathbb{C}^2)^{\oplus 6}]^{Sl_2}$, generated by the projections $p_i\colon (\mathbb{C}^2)^{\oplus 6} \to \mathbb{C}^2$ to the i-th component.
The restrictions $p_i|_{\mu^{-1}(0)}\colon \mu^{-1}(0) \to \mathbb{C}^2$ still span a 6-dimensional space: Consider for example the matrices M_i where each except the i-th column is 0. Then $M_i Q M_i^t = 0$ for $i = 1, \dots, 6$, so $M_i \in \mu^{-1}(0)$. In turn $p_j(M_i) = \delta_{ij}\binom{x_{1j}}{x_{2j}}$ shows that the $p_i|_{\mu^{-1}(0)}$ are linearly independent. Thus $\mathrm{Mor}_{Sl_2}(\mu^{-1}(0), \mathbb{C}^2) \cong \mathrm{Hom}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$ and $F_1 =< p_i \mid i = 1, \dots, 6 >\cong \mathbb{C}^6$. The SO_6-equivariant identification $\mathbb{C}^6 \cong \mathrm{Hom}_{Sl_2}((\mathbb{C}^2)^{\oplus 6}, \mathbb{C}^2)$, $e_i \mapsto p_i$ induces the inner product $\langle p_i, p_j \rangle = \delta_{i+3,j} + \delta_{j+3,i}$ on $< p_1, \dots p_6 >$. For this reason we can also write $\langle p, q \rangle = p^t Q q$ for all maps $p, q \in F_1$ and we see that F_1 is the standard representation. □

4.2. **Embedding the Hilbert scheme into a product of Grassmannians.** As remarked in the proof of proposition 4.2, every map $S \to \mathrm{Hilb}_h^G(X)$ gives us a map $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \to \mathcal{F}_\rho$ by pulling back (3). Since $\mathcal{F}_\rho$ is a locally free quotient of $\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho$ of rank $h(\rho)$, this in turn corresponds to a map $S \to \mathrm{Grass}(F_\rho, h(\rho))$ into the Grassmannian of quotients of F_ρ of dimension $h(\rho)$. In particular, taking $S = \mathrm{Hilb}_h^G(X)$, we obtain a map of schemes

$$\eta_\rho\colon\ \mathrm{Hilb}_h^G(X) \to \mathrm{Grass}(F_\rho, h(\rho)).$$

In the situation of proposition 4.2 this map is again H-equivariant. Evaluating at a closed point $s \in S$ yields

$$\begin{array}{lcccc} (S \to \mathrm{Hilb}_h^G(X)) & \longmapsto & (\mathcal{O}_S \otimes_{\mathbb{C}} F_\rho \to \mathcal{F}_\rho) & \longmapsto & (S \to \mathrm{Grass}(F_\rho, h(\rho))), \\ (s \mapsto Z) & \longmapsto & (f_{\rho,s}\colon F_\rho \to \mathcal{F}_\rho(s)) & \longmapsto & (s \mapsto \mathcal{F}_\rho(s)), \end{array} \tag{4}$$

where the fibres $\mathcal{F}_\rho(s)$ are vector spaces of dimension $h(\rho)$. Hence for $S = \mathrm{Hilb}_h^G(X)$ we have

$$\eta_\rho\colon\ \mathrm{Hilb}_h^G(X) \to \mathrm{Grass}(F_\rho, h(\rho)),\ Z \mapsto \mathcal{F}_\rho(Z).$$

As $\mathbb{C}[X]_\rho = \mathrm{Hom}_G(V(\rho), \mathbb{C}[X]) \cong \mathrm{Mor}_G(X, V(\rho)^*)$, the elements of the generating space F_ρ are G-equivariant morphisms from X to $V(\rho)^*$ and evaluating at an element $Z \in \mathrm{Hilb}_h^G(X)$ means restricting $\mathrm{Mor}_G(X, V(\rho)^*) \to \mathrm{Mor}_G(Z, V(\rho)^*)$, so in (4) we have

$$f_{\rho,Z}\colon F_\rho \twoheadrightarrow \mathcal{F}_\rho(Z),\ p \mapsto p|_Z.$$

The map η_{ρ_0} does not yield any information because $\mathrm{Grass}(F_{\rho_0}, h(\rho_0)) = \mathrm{Grass}(\mathbb{C}, 1)$ is only a point. The product of the Hilbert-Chow morphism and the η_ρ defines a map

$$\mathrm{Hilb}_h^G(X) \to X/\!\!/G \times \prod_{\substack{\rho\in\mathrm{Irr}(G)\\ \rho\neq 0}} \mathrm{Grass}(F_\rho, h(\rho)). \tag{5}$$

This map is a closed immersion, even if the product ranges over an appropriately chosen finite subset of $\mathrm{Irr}(G)$: Indeed, let $B = TU$ be a Borel subgroup of G, where T is a maximal torus and U the unipotent radical. Assigning to $V(\rho)$ its highest weight gives a one-to-one correspondence between $\mathrm{Irr}\, G$ and the set of dominant weights Λ^+ in the weight lattice Λ of T. Extend h to Λ by 0. Let V be a finite-dimensional T-module containing $X/\!\!/U$. By [AB05, Thm 1.7, Lemma 1.6], we have closed embeddings $\mathrm{Hilb}_h^G(X) \hookrightarrow \mathrm{Hilb}_h^T(X/\!\!/U) \hookrightarrow \mathrm{Hilb}_h^T(V)$ and each module $\mathbb{C}[V]_\rho$ is generated by some $\mathbb{C}$-vector space E_ρ over $\mathbb{C}[V]^G$. The E_ρ can be chosen as lifts of F_ρ, so that we have $E_\rho \twoheadrightarrow F_\rho$ under $\mathbb{C}[V] \twoheadrightarrow \mathbb{C}[X]$. As shown by [HS04, Thm 2.2, 2.3] the map

$$\mathrm{Hilb}_h^T(V) \hookrightarrow \prod_{\rho\in D} \mathrm{Grass}(E_\rho, h(\rho))$$

is a closed immersion for an appropriately chosen finite subset $D \subset \Lambda$. Since $h = 0$ outside Λ^+ we even have $D \subset \mathrm{Irr}(G)$ in our case. Every quotient of F_ρ of dimension $h(\rho)$ is also a quotient of E_ρ of dimension $h(\rho)$, so we have an embedding $\mathrm{Grass}(F_\rho, h(\rho)) \hookrightarrow \mathrm{Grass}(E_\rho, h(\rho))$. As every element in $\mathrm{Hilb}_h^T(V)$ coming from $\mathrm{Hilb}_h^T(X/\!\!/U)$ is already generated by F_ρ, the composite map $\mathrm{Hilb}_h^T(X/\!\!/U) \hookrightarrow \prod_{\rho\in D} \mathrm{Grass}(E_\rho, h(\rho))$ factors through $\mathrm{Grass}(F_\rho, h(\rho))$, so that we obtain

$$\begin{tikzcd}
 & \prod\limits_{\rho\in D} \mathrm{Grass}(F_\rho, h(\rho)) \arrow[r, hook] & \prod\limits_{\rho\in D} \mathrm{Grass}(E_\rho, h(\rho)) \\
\mathrm{Hilb}_h^G(X) \arrow[r, hook] \arrow[ur, hook, dashed] \arrow[d] & \mathrm{Hilb}_h^T(X/\!\!/U) \arrow[r, hook] \arrow[u, hook, dashed] \arrow[d] & \mathrm{Hilb}_h^T(V) \arrow[u, hook] \arrow[d] \\
X/\!\!/G \arrow[r, equal] & (X/\!\!/U)/\!\!/T \arrow[r, hook] & V/\!\!/T
\end{tikzcd}$$

This suggests the following procedure to determine the invariant Hilbert scheme: Begin with one "easy" representation ρ_i and analyse $\eta \times \eta_{\rho_i}$. If this can be shown to be closed immersion, identify the image. Otherwise add another representation and repeat the analysis. This process will stop with some $\eta \times \eta_{\rho_1} \times \ldots \times \eta_{\rho_s}$ being closed immersion.

Determination of η_1. The knowledge of F_1 gives us an SO_6-equivariant map

$$\eta_1\colon Sl_2\text{-}\mathrm{Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(F_1, \dim V_1) = \mathrm{Grass}(\mathbb{C}^6, 2),\ Z \mapsto \mathcal{F}_1(Z).$$

The fibre $\mathcal{F}_1(Z)$ of the sheaf $\mathcal{F}_1$ is generated by the restrictions of the projections $p_i\colon \mu^{-1}(0) \to \mathbb{C}^2$ to the subscheme $Z \subset \mu^{-1}(0)$.

Proposition 4.4. (1) *The map $\eta \times \eta_1$ is given by*

$$\eta \times \eta_1 \colon Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mu^{-1}(0)/\!\!/Sl_2 \times \mathrm{Grass}(2, \mathbb{C}^6),\ Z \mapsto (Z/\!\!/Sl_2, \ker(f_{1,Z})^{\perp}).$$

(2) *The image of $\eta \times \eta_1$ restricted to the orbit component $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ is contained in $Y := \{(A, U) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset U\}$.*

Proof 1. To describe the morphism $\eta_1 \colon Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(\mathbb{C}^6, 2)$ explicitly, we analyse the map $f_{1,Z} \colon F_1 \to \mathcal{F}_1(Z)$. As it is surjective, we have $\mathcal{F}_1(Z) \cong F_1/\ker(f_{1,Z})$. Now we can identify the Grassmannian of quotients with the Grassmannian of subspaces via the canonical isomorphism $\mathrm{Grass}(\mathbb{C}^6, 2) \to \mathrm{Grass}(2, \mathbb{C}^6)$, $F_1/\ker(f_{1,Z}) \mapsto \ker(f_{1,Z})^{\perp}$. Thus η_1 is the morphism $\eta_1 \colon Sl_2\text{-Hilb}(\mu^{-1}(0)) \to \mathrm{Grass}(2, \mathbb{C}^6)$, $Z \mapsto \ker(f_{1,Z})^{\perp}$.
2. Over $\mathcal{O}_{[2^2,1^2]}$, we have $\eta \times \eta_1 : \eta^{-1}(\mathcal{O}_{[2^2,1^2]}) \to \mathcal{O}_{[2^2,1^2]} \times \mathrm{Grass}(2, \mathbb{C}^6)$, $Z_A \mapsto (A, \ker(f_{1,Z_A})^{\perp})$. For analysing the image, we choose the special point $A_0 \in \mathcal{O}_{[2^2,1^2]}$ again. The description above shows that $\ker(f_{1,Z_{A_0}}) = < p_3, p_4, p_5, p_6 >$ with orthogonal complement $\ker(f_{1,Z_{A_0}})^{\perp} = < p_4, p_5 >$ by definition of the inner product above. Since $p_4^t Q p_4 = p_4^t Q p_5 = p_5^t Q p_5 = 0$, this space is isotropic. Thus for every point A in the open orbit, $\ker(f_{1,Z_A})^{\perp}$ is isotropic. As being isotropic is a closed condition, $\eta \times \eta_1$ maps the closure of the preimage of $\mathcal{O}_{[2^2,1^2]}$ under η, the orbit component, to the isotropic Grassmannian:

$$\eta \times \eta_1 \colon \overline{\eta^{-1}(\mathcal{O}_{[2^2,1^2]})} = Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \to \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6).$$

For the additional condition, we only need to examine $A_0 = \begin{pmatrix} 0 & \begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{pmatrix}$ again. We can consider A_0 and its transpose A_0^t as maps

$$A_0 \colon F_1 \to F_1,\ p_4 \to -p_2,\ p_5 \to p_1,\ p_i \to 0 \text{ for } i = 1, 2, 3, 6,$$
$$A_0^t \colon F_1 \to F_1,\ p_1 \to p_5,\ p_2 \to -p_4,\ p_i \to 0 \text{ for } i = 3, 4, 5, 6.$$

Thus we have $\mathrm{im}(A_0^t) = < p_4, p_5 > = \ker(f_{1,Z_{A_0}})^{\perp}$. Since $\eta \times \eta_1$ is SO_6-equivariant, the equality $\mathrm{im}(A^t) = \ker(f_{1,Z_A})^{\perp}$ holds for every A in the orbit $\mathcal{O}_{[2^2,1^2]}$ and we obtain

$$\eta \times \eta_1(\eta^{-1}(\mathcal{O}_{[2^2,1^2]})) \subset Y' := \{(A, U) \in \mathcal{O}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t = U\}.$$

If $A \in \overline{\mathcal{O}}_{[2^2,1^2]} \setminus \mathcal{O}_{[2^2,1^2]}$, its rank is smaller than 2 (indeed $A = 0$), and so is $\dim(\mathrm{im}\, A^t)$. So the closure of Y' in $\overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6)$ is Y. □
We will see in the further examination that $\eta \times \eta_1$ actually is an isomorphism (proposition 4.6), even on the whole invariant Hilbert scheme (proposition 5.4).

4.3. **The Grassmannian as a homogeneous space.** For a further analysis of the image we consider the isotropic Grassmannian as a homogeneous space $\mathrm{Grass}_{iso}(2, \mathbb{C}^6) = SO_6/P$, where $P = (SO_6)_{W_0}$ is the isotropy group of an arbitrary point $W_0 \in \mathrm{Grass}_{iso}(2, \mathbb{C}^6)$. We choose $W_0 = < p_1, p_2 >$. If $g_W \in SO_6$ is chosen such that $W = g_W W_0$, the isomorphism is

$$\mathrm{Grass}_{iso}(2, \mathbb{C}^6) \to SO_6/P,\ W \mapsto g_W P = [g_W],\ gW_0 \leftarrow\!\shortmid [g].$$

The existence of the canonical map $f \colon Y \xrightarrow{pr_2} \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \cong SO_6/P$, $(A, U) \mapsto U \mapsto [g_U]$ shows that Y is an associated SO_6-bundle with fibre $E := f^{-1}([I_6]) = pr_2^{-1}(W_0)$:

$$\begin{tikzcd}
(g,A)P \arrow[rr, mapsto] \arrow[dr, mapsto] & SO_6 \times^P E \arrow[rr, "\cong"] \arrow[dr] & (gA, gW_0) & Y \arrow[dl, "f"] & \ni (A,W) \arrow[dl, mapsto] \\
 & {[g] = gP} & SO_6/P & \ni g_W P &
\end{tikzcd}$$

where $SO_6 \times^P E = SO_6 \times E/\sim$ with $(g, A) \sim (gp^{-1}, pAp^{-1})$.

Lemma 4.5. *The fibre* $E = \{A \in \overline{\mathcal{O}}_{[2^2,1^2]} \mid \operatorname{im} A^t \subset W_0\}$ *is one-dimensional.*

Proof Let $A^t = (a_{ij})$, i.e. $A^t p_i = \sum a_{ji} p_j$. We have

- $\operatorname{im} A^t \subset W_0 = < p_1, p_2 >$, thus $a_{ij} = 0$ if $i = 3, 4, 5, 6$,
- by duality, $W_0^\perp = < p_1, p_2, p_3, p_6 > \subset \ker A^t$, which implies $a_{ij} = 0$ if $j = 1, 2, 3, 6$.

There only remain a_{14}, a_{24}, a_{15} and a_{25}. But

- $A^t \in \mathfrak{so}_6$ implies $a_{14} = a_{25} = 0$ and $a_{24} = -a_{15}$.

Thus E is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. $\square$

Connecting this to the Hilbert scheme, we have

$$\begin{tikzcd}
& \mu^{-1}(0)/\!\!/Sl_2 & \\
Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \arrow[ur, "\eta"] \arrow[rr, "\eta\times\eta_1"] \arrow[dr, "f'=f\circ(\eta\times\eta_1)"] & & Y \cong SO_6 \times^P E \arrow[ul, "pr_1"'] \arrow[dl, "f"] \\
& SO_6/P &
\end{tikzcd}$$

The existence of f' shows, that $Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}$ can be written as an associated SO_6-bundle with fibre $F := f'^{-1}([I_6])$ and combining the two SO_6-bundles we obtain

$$\begin{tikzcd}
SO_6 \times^P F \arrow[rr, "\cong"] \arrow[d, "(\eta\times\eta_1)'"'] \arrow[ddr] & & Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \arrow[d, "\eta\times\eta_1"] \arrow[ddl, "f'"'] \\
SO_6 \times^P E \arrow[rr, "\cong"] \arrow[dr] & & Y \arrow[dl, "f"] \\
& SO_6/P &
\end{tikzcd}$$

As $\eta \times \eta_1$ is birational and proper, restricting $(\eta \times \eta_1)'$ to the fibre over a fixed point of SO_6 yields a birational and proper morphism $\psi\colon F \to E$. Since E is isomorphic to the affine line, ψ must be an isomorphism. As a consequence:

Proposition 4.6. *The orbit component of the Sl_2-Hilbert scheme is isomorphic to Y:*

$$Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} \cong \{(A, U) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \operatorname{Grass}_{iso}(2, \mathbb{C}^6) \mid \operatorname{im} A^t \subset W\}.$$

4.4. **The points of $\operatorname{Hilb}^G_h(X)^{orb}$ as subschemes of X.** To identify the points of $\operatorname{Hilb}^G_h(X)^{orb}$ as subschemes of X, we assume there is an embedding

$$\operatorname{Hilb}^G_h(X)^{orb} \hookrightarrow X/\!\!/G \times \prod_{\rho \in M} \operatorname{Grass}(F_\rho, h(\rho)),\ Z \to (Z/\!\!/G, (\mathcal{F}_\rho(Z))_{\rho \in M})$$

where $M \subset \operatorname{Irr} G$ is a suitable finite subset and $\mathcal{F}_\rho(Z) = F_\rho / \ker f_{\rho,Z}$ with $f_{\rho,Z}\colon F_\rho \to \mathcal{F}_\rho(Z)$. This embedding gives us the invariant part and the ρ-parts of the ideal I_Z of Z as

$$(I_Z)^G = I_{Z/\!\!/G}$$
$$(I_Z)_\rho = (\ker f_{\rho,Z}).$$

Thus $I_Z \supset I_M := < I_{Z/\!\!/G}, \ker f_{\rho,Z} \mid \rho \in M >$. If I_M already has Hilbert function h, then I_Z has no further generators and we obtain $I_Z = I_M$.

The points of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ as subschemes of $\mu^{-1}(0)$. Our next goal is to describe the points (A,W) of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ as subschemes of $\mu^{-1}(0)$.

Proposition 4.7. *The subscheme $Z_{A,W} \subset \mu^{-1}(0)$ corresponding to $(A,W) \in Y$ is*

$$Z_{A,W} \cong \begin{cases} Sl_2, & \text{if } A \in \mathcal{O}_{[2^2,1^2]}, \\ \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \middle| ad-bc=0 \right\}, & \text{if } A = 0. \end{cases}$$

Proof To show this, we use the embedding $\eta \times \eta_1 \colon Sl_2$-Hilb$(\mu^{-1}(0))^{orb} \to \mu^{-1}(0)/\!\!/Sl_2 \times \mathrm{Grass}(2,\mathbb{C}^6)$, $Z \mapsto (Z/\!\!/Sl_2, \ker(f_{1,Z})^{\perp})$ and we have to compute $Z_{A,W} = (\eta\times\eta_1)^{-1}(A,W)$ or its ideal $I_{A,W}$. The action of SO_6 on the Hilbert scheme and on Y reduces this to the calculation of one $Z_{A,W}$ for every orbit of Y: Since $\eta \times \eta_1$ is SO_6-equivariant, all points in the preimage of one orbit are isomorphic. Y decomposes into two SO_6-orbits $\{(A, \operatorname{im} A^t) \mid A \in \mathcal{O}_{[2^2,1^2]}\} \cong \mathcal{O}_{[2^2,1^2]}$ and $\{0\} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6)$, because the action on $\mathrm{Grass}_{iso}(2,\mathbb{C}^6)$ is transitive.
First we consider $A \in \mathcal{O}_{[2^2,1^2]}$. Since η is an isomorphism of schemes over $\mathcal{O}_{[2^2,1^2]}$, we already know that $Z_{A,W} = \eta^{-1}(A) = \nu^{-1}(A) \cong Sl_2$ by section 3.2.
Now let $A \in \overline{\mathcal{O}_{[2^2,1^2]}} \setminus \mathcal{O}_{[2^2,1^2]} = \{0\}$. Then $Z_{0,W}/\!\!/Sl_2 = 0$, so all 2×2-minors of elements in $Z_{0,W}$ vanish, i.e. $(I_{0,W})^{Sl_2} = (\Lambda_{ij} \mid i,j = 1,\dots,6)$. We calculate the subscheme $Z_{0,W}$ explicitly for $W = W_0 :=< p_1, p_2 >$. Consider $f_{1,Z_{0,W_0}} \colon F_1 \to \mathcal{F}_1(Z_{0,W_0})$, $q \mapsto q|_{Z_{0,W_0}}$. We know that $W_0 = \ker(f_{1,Z_{0,W_0}})^{\perp}$. If $q = \sum_{i=1}^6 a_i p_i \in \ker(f_{1,Z_{0,W_0}})$, we have $0 = q(M) = \sum_{i=1}^6 a_i \left(\begin{smallmatrix} x_{1i} \\ x_{2i} \end{smallmatrix}\right)$ for every $M \in Z_{0,W_0}$. Thus the component of I_{0,W_0} corresponding to the standard representation is $(I_{0,W_0})_1 = (\sum_{i=1}^6 a_i x_{1i},\ \sum_{i=1}^6 a_i x_{2i} \mid q \in W_0^{\perp})$ and for the induces subscheme $Z'_{0,W_0} := \mathrm{Spec}(\mathbb{C}[\mu^{-1}(0)]/((I_{0,W_0})^{Sl_2} + (I_{0,W_0})_1)) \supset Z_{0,W_0}$ we have

$$Z'_{0,W_0} = \{M \in (\mathbb{C}^2)^{\oplus 6} \mid MQM^t = 0, \Lambda^{ij} = 0\ \forall\, i,j, \sum_{i=1}^{6} a_i x_{1i} = 0 = \sum_{i=1}^{6} a_i x_{2i}\ \forall\, q \in W_0^{\perp}\}.$$

In our case, $W_0^{\perp} = <p_1, p_2, p_3, p_6>$, thus letting q be each of these generators yields the equations $x_{1i} = 0 = x_{2i}$ if $i = 1,2,3,6$. This means that M takes the shape $M = \left(\begin{smallmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{smallmatrix}\right)$ and $0 = \Lambda^{45} = x_{14}x_{25} - x_{15}x_{24}$. Then the equation $MQM^t = 0$ is automatically fulfilled. So we obtain

$$Z'_{0,W_0} = \left\{ \begin{pmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{pmatrix} \in (\mathbb{C}^2)^{\oplus 6} \middle| x_{14}x_{25} - x_{15}x_{24} = 0 \right\}.$$

Since this is a flat deformation of Sl_2, the corresponding ideal has the correct Hilbert function, which means that $I_{0,W_0} = ((I_{0,W_0})^{Sl_2} + (I_{0,W_0})_1)$ and $Z_{0,W_0} = Z'_{0,W_0}$. □

5. Properties of the invariant Hilbert scheme

To determine the whole Sl_2-Hilbert scheme, we analyse some of its properties. First, we show that Sl_2-Hilb$(\mu^{-1}(0))$ is smooth in every point of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ in section 5.1, so that the orbit component is a smooth connected component of Sl_2-Hilb$(\mu^{-1}(0))$. Section 5.2 concludes the proof of theorem 1.1, namely Sl_2-Hilb$(\mu^{-1}(0)) = \{(A,W) \in \overline{\mathcal{O}}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2,\mathbb{C}^6) \mid \operatorname{im} A^t \subset W\}$, by showing that Sl_2-Hilb$(\mu^{-1}(0))$ is connected and hence coincides with the orbit component.

5.1. **Smoothness.** One way to examine smoothness of the Hilbert scheme is to calculate its tangent space at every point. In smooth points the dimension of the tangent space equals the dimension of the Hilbert scheme, in singular points the dimension of the tangent space is bigger. In the first case, one concludes that the orbit component is smooth and that there is no additional component of the invariant Hilbert scheme intersecting it, so $\mathrm{Hilb}_h^G(X)^{orb}$ is a connected component of the invariant Hilbert scheme.

Let $Z \in \mathrm{Hilb}_h^G(X)$, $R := \Gamma(X, \mathcal{O}_X)$ and $\mathcal{I}_Z$ the ideal of Z in $\mathcal{O}_X$ with space of global sections I_Z.

Proposition 5.1. [AB05, § 1.4] *The tangent space of the Hilbert scheme is given by*

$$T_Z \mathrm{Hilb}_h^G(X) = Hom_R^G(I_Z, R/I_Z) = Hom_{R/I_Z}^G(I_Z/I_Z^2, R/I_Z) = H^0(\mathcal{H}om_{\mathcal{O}_Z}^G(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)).$$

Remark 5.2. In special situations more can be said about the structure of the tangent space:

(1) If Z is smooth and contained in the regular part X_{reg} of X, then the normal sheaf $\mathcal{N}_{Z/X} := (\mathcal{I}_Z/\mathcal{I}_Z^2)^\vee = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z)$ is locally free. This yields a further description of the tangent space

$$T_Z \mathrm{Hilb}_h^G(X) = Hom_{R/I_Z}^G(I_Z/I_Z^2, R/I_Z) = H^0(Z, \mathcal{N}_{Z/X})^G.$$

(2) If $Z = Gx \cong G$ is an orbit isomorphic to the group, then we have a commutative diagram

$$\begin{tikzcd} G \times \mathcal{N}_{Z/X,e} \arrow[r, "\sigma", "\text{action}"'] \arrow[dr, "\text{projection}"'] & \mathcal{N}_{Z/X} \arrow[d] \\ & G \cong Z \end{tikzcd}$$

where $\mathcal{N}_{Z/X,e}$ is the fibre of $\mathcal{N}_{Z/X}$ at the neutral element $e \in G$. The action σ restricted to $G \times \mathcal{N}_{Z/X,e}$ is an isomorphism on the fibres: This is clear for $\mathcal{N}_{Z/X,e}$ and true for the other fibres since σ is G-equivariant. Since both spaces are vector bundles and σ is linear, they are isomorphic.

Giving a G-invariant section $s\colon G \to G \times \mathcal{N}_{Z/X,e}$ means choosing a point $p \in \mathcal{N}_{Z/X,e}$ such that $s = \mathrm{id} \times p$. This shows

$$T_Z \mathrm{Hilb}_h^G(X) = H^0(Z, \mathcal{N}_{Z/X})^G \cong \mathcal{N}_{Z/X,e}.$$

(3) If Z is not smooth we can consider its regular part Z_{reg}. If Z is reduced, restricting morphisms to Z_{reg} yields injections

$$Hom_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) \hookrightarrow Hom_{\mathcal{O}_{Z_{reg}}}(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}})$$

and

$$Hom_{\mathcal{O}_Z}^G(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) \hookrightarrow Hom_{\mathcal{O}_{Z_{reg}}}^G(\mathcal{I}_{Z_{reg}}/\mathcal{I}_{Z_{reg}}^2, \mathcal{O}_{Z_{reg}}),$$

If $Z_{reg} \subset X_{reg}$, taking global sections we obtain

$$Hom_{R/I_Z}^G(I_Z/I_Z^2, R/I_Z) \hookrightarrow H^0(Z_{reg}, \mathcal{N}_{Z_{reg}/X_{reg}})^G.$$

All these maps are isomorphisms if Z is normal.

The tangent space of Sl_2-Hilb$(\mu^{-1}(0))$. In order to find out if the orbit component coincides with the whole Hilbert scheme, we calculate the tangent space to Sl_2-Hilb$(\mu^{-1}(0))$ in every point of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$.

Proposition 5.3. *For every point $Z \in Sl_2$-Hilb$(\mu^{-1}(0))^{orb}$ the dimension of the tangent space is*

$$\dim T_Z Sl_2\text{-Hilb}(\mu^{-1}(0)) = 6 = \dim Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb}.$$

Therefore the orbit component is a smooth connected component of the invariant Hilbert scheme.

Proof As before, we only have to consider one point of each SO_6-orbit because the dimension of the tangent space is stable in every orbit of the SO_6-action. Over the open orbit there is nothing to show, because we know that $\eta^{-1}(\mathcal{O}_{[2^2,1^2]}) \cong \mathcal{O}_{[2^2,1^2]}$ is smooth.

Over the origin we consider

$$Z := Z_{0,W_0} = \left\{ \begin{pmatrix} 0 & 0 & 0 & x_{14} & x_{15} & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & 0 \end{pmatrix} \middle| x_{14}x_{25} - x_{15}x_{24} = 0 \right\}$$
$$\cong \left\{ \begin{pmatrix} \lambda x & \lambda y \\ \mu x & \mu y \end{pmatrix} \middle| x, y \in \mathbb{C}, [\lambda : \mu] \in \mathbb{P}^1 \right\}.$$

Z is normal since it is a complete intersection and the codimension of $Z \setminus \dot{Z} = \{0\}$ in Z is greater than 2, namely 3. We have $Z \subset \mu^{-1}(0)_{sing}$: If $M \in Z$, all of its 2×2-minors vanish, thus $M \in V(X^t JX) = \mu^{-1}(0)_{sing}$, where $X = \left(\begin{smallmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \end{smallmatrix}\right)$ describes the coordinates in $\mathbb{C}[x_{11}, \dots, x_{26}]$.

From now on we also write $a := x_{14}$, $b := x_{15}$, $c := x_{24}$ and $d := x_{25}$. Let $\mathcal{I}$ be the ideal of Z in $R := \mathbb{C}[\mu^{-1}(0)] = \mathbb{C}[x_{11}, \dots, x_{26}]/(XQX^t)$. We have

$$\mathcal{I} = (x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, \underbrace{x_{14}x_{25} - x_{15}x_{24}}_{=:z}),$$
$$R/\mathcal{I} = \mathbb{C}[x_{11}, \dots, x_{26}]/(x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, x_{14}x_{25} - x_{15}x_{24}, XQX^t)$$
$$= \mathbb{C}[a, b, c, d]/(ad - bc).$$

Then $\mathcal{I}/\mathcal{I}^2 = R < x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, z >$ with relations $XQX^t = 0$:

$$\begin{aligned} 0 &= x_{11}x_{14} + x_{12}x_{15} + x_{13}x_{16} \equiv x_{11}a + x_{12}b \\ 0 &= x_{11}x_{24} + x_{12}x_{25} + x_{13}x_{26} + x_{14}x_{21} + x_{15}x_{22} + x_{16}x_{23} \equiv x_{11}c + x_{12}d + x_{21}a + x_{22}b \\ 0 &= x_{21}x_{24} + x_{22}x_{25} + x_{23}x_{26} \equiv x_{21}c + x_{22}d \qquad mod \mathcal{I}^2. \end{aligned}$$

Reduction to Z_{reg}

We analyse the tangent space $T_Z Sl_2\text{-Hilb}(\mu^{-1}(0))$ of the invariant Hilbert scheme by reducing to

$$\dot{Z} := Z_{reg} = Z \setminus \{0\} = \{(\lambda v, \mu v) \mid v \in \mathbb{C}^2 \setminus \{0\}, [\lambda : \mu] \in \mathbb{P}^1\}.$$

Let $\dot{\mathcal{I}}$ be the ideal sheaf of $\dot{Z}$. Then in the situation of remark 5.2(3) we even have equality because Z is normal. Thus

$$\dim T_Z Sl_2\text{-Hilb}(\mu^{-1}(0)) = H^0(Z, \mathcal{H}om^G_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z))^{Sl_2} = \dim H^0(\dot{Z}, \mathcal{H}om^G_{\mathcal{O}_{\dot{Z}}}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2}.$$

The open subscheme $\dot{Z}$ of Z is not affine. Since $\dot{Z} = Z \setminus \{0\} = V(ad - bc) \setminus V(a, b, c, d)$, it is covered by the open affine sets $\dot{Z}_a = \operatorname{Spec} R_a$, $\dot{Z}_b = \operatorname{Spec} R_b$, $\dot{Z}_c = \operatorname{Spec} R_c$ and $\dot{Z}_d = \operatorname{Spec} R_d$, where

$$\begin{aligned} R_a &= (\mathbb{C}[a, b, c, d]/(ad - bc))_a = \mathbb{C}[a, a^{-1}, b, c, d]/(ad - bc) = \mathbb{C}[a, a^{-1}, b, c], \\ &\qquad \text{since } a \text{ is invertible and } d = \frac{bc}{a}, \\ R_b &= (\mathbb{C}[a, b, c, d]/(ad - bc))_b = \mathbb{C}[a, b, b^{-1}, d], \\ R_c &= (\mathbb{C}[a, b, c, d]/(ad - bc))_c = \mathbb{C}[a, c, c^{-1}, d], \\ R_d &= (\mathbb{C}[a, b, c, d]/(ad - bc))_d = \mathbb{C}[b, c, d, d^{-1}]. \end{aligned}$$

To describe the ideal sheaf $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$, we compute it on each set of this covering. As $\dot{\mathcal{I}} = \mathcal{I}|_{\dot{Z}}$ and $\mathcal{I}$ conincide on an open subset, $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ is generated by $x_{11}, x_{12}, x_{13}, x_{16}, x_{21}, x_{22}, x_{23}, x_{26}, z$ with relations

$$\begin{aligned} 0 &= x_{11}a + x_{12}b \\ 0 &= x_{11}c + x_{12}d + x_{21}a + x_{22}b \\ 0 &= x_{21}c + x_{22}d. \end{aligned}$$

Since a is invertible in R_a, the first relation yields $x_{11} = -\frac{b}{a}x_{12}$. The second relation becomes $0 = -\frac{b}{a}x_{12}c + x_{12}\frac{bc}{a} + x_{21}a + x_{22}b = x_{21}a + x_{22}b$, thus $x_{21} = -\frac{b}{a}x_{22}$. Then the third equation $0 = -\frac{b}{a}x_{22}c + x_{22}\frac{bc}{a}$ is automatically fulfilled and gives no more information. Denoting $\dot{\mathcal{I}}_a := \dot{\mathcal{I}}|_{\dot{Z}_a}$, this shows that

$$\dot{\mathcal{I}}_a/\dot{\mathcal{I}}_a^2 = R_a < x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z >$$

is free of rank 7. This means that $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ is locally free of rank 7, since we obtain analogously

$$\begin{aligned}
\dot{\mathcal{I}}_b/\dot{\mathcal{I}}_b^2 &= R_b < x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z >,\\
\dot{\mathcal{I}}_c/\dot{\mathcal{I}}_c^2 &= R_c < x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z >,\\
\dot{\mathcal{I}}_d/\dot{\mathcal{I}}_d^2 &= R_d < x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z > .
\end{aligned}$$

If we compute the intersection $\dot{Z}_{ab} = \operatorname{Spec} R_{ab}$ of $\dot{Z}_a$ and $\dot{Z}_b$, we obtain

$$\begin{aligned}
R_{ab} &= \mathbb{C}[a, a^{-1}, b, b^{-1}, c, d]/(ad - bc) = \mathbb{C}[a, a^{-1}, b, b^{-1}, c] = \mathbb{C}[a, a^{-1}, b, b^{-1}, d],\\
\dot{\mathcal{I}}_{ab}/\dot{\mathcal{I}}_{ab}^2 &= \mathbb{C}[a, a^{-1}, b, b^{-1}, c] < x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z >\\
&= \mathbb{C}[a, a^{-1}, b, b^{-1}, d] < x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z >
\end{aligned}$$

with $d = \frac{b}{a}c$ and base change $x_{11} = -\frac{b}{a}x_{12}$ and $x_{21} = -\frac{b}{a}x_{22}$.

Reduction of Sl_2-linearised sheaves to sheaves linearised w.r.t. a Borel subgroup
To compute $H^0(\dot{Z}, \mathcal{H}om(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2}$, we reduce the Sl_2-linearised sheaf $\dot{\mathcal{I}}/\dot{\mathcal{I}}^2$ on $\dot{Z}$ to a B-linearised sheaf on $\mathbb{C}^2 \setminus \{0\}$, where B is the Borel subgroup of upper triangular matrices of Sl_2.

Claim. *$\dot{Z}$ is an associated B-bundle:*

$$\dot{Z} = Sl_2 \times^B E, \quad \textit{where} \quad E = \pi^{-1}(e_1) = \{(\lambda e_1, \mu e_1) \mid [\lambda : \mu] \in \mathbb{P}^1\} \cong \mathbb{C}^2 \setminus \{0\}.$$

Proof There is a natural map

$$\varphi\colon \dot{Z} \to \mathbb{P}^1 \times \mathbb{P}^1,\ (\lambda v, \mu v) \mapsto ([v], [\lambda : \mu]).$$

Since $g \cdot (\lambda v, \mu v) = (\lambda gv, \mu gv)$ for every $g \in Sl_2$, φ is equivariant for the action $g \cdot ([v], [\lambda : \mu]) = ([gv], [\lambda : \mu])$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This yields an equivariant projection

$$\pi\colon \dot{Z} \to \mathbb{P}^1,\ (\lambda v, \mu v) \mapsto [v].$$

As the action of Sl_2 on $\mathbb{P}^1$ is transitive with isotropy group $B = \left\{\left(\begin{smallmatrix} t & u \\ 0 & t^{-1}\end{smallmatrix}\right) \middle| \, t \in \mathbb{C}^*, u \in \mathbb{C}\right\}$, we can write $\mathbb{P}^1 = Sl_2/B$. If $gB \in Sl_2/B$ and $b = \left(\begin{smallmatrix} t & u \\ 0 & t^{-1}\end{smallmatrix}\right)$, we have

$$gb = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} tg_{11} & ug_{11} + t^{-1}g_{12} \\ tg_{21} & ug_{21} + t^{-1}g_{22} \end{pmatrix},$$

which shows that in the class of g, g_{11} and g_{21} are determined up to a scalar and g_{12} and g_{22} can be modified arbitrarily up to the condition $\det(g) = 1$. Therefore the identification $Sl_2/B \cong \mathbb{P}^1$ is $gB \mapsto [g_{11} : g_{21}]$.

The action of B on $\mathbb{C}^2 \setminus \{0\}$ induced by the action of Sl_2 on $\dot{Z}$ can be computed as follows: Let $b = \left(\begin{smallmatrix} t & u \\ 0 & t^{-1}\end{smallmatrix}\right) \in B$. Then $be_1 = \left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right) = te_1$, thus $b(\lambda e_1, \mu e_1) = (t\lambda e_1, t\mu e_1)$ and this means

$$b \cdot (\lambda, \mu) = (t\lambda, t\mu).$$

Hence the action of B on $\mathbb{C}^2 \setminus \{0\}$ coincides with the action of $\mathbb{C}^*$.

Altogether, we have the following commutative diagram

$$\begin{tikzcd}
 & (g,(\lambda,\mu)) \arrow[r, mapsto] & \left(\lambda\binom{g_{11}}{g_{21}},\mu\binom{g_{11}}{g_{21}}\right) & \\
(g,(\lambda,\mu)) \arrow[d, mapsto] & Sl_2 \times^B \mathbb{C}^2 \setminus \{0\} \arrow[r, "\cong"] \arrow[d] & \dot{Z} \arrow[d] & (\lambda v,\mu v) \arrow[d, mapsto] \\
gB & Sl_2/B \arrow[r, "\cong"] & \mathbb{P}^1 & [v] \\
 & gB \arrow[r, mapsto] & [g_{11}:g_{21}] &
\end{tikzcd}$$

⊠

Now an Sl_2-linearised sheaf $\mathcal{F}$ on $\dot{Z}$ corresponds to a B-linearised sheaf $\mathcal{G}$ on $\mathbb{C}^2 \setminus \{0\}$ as well as their duals correspond to each other. If $j\colon \mathbb{C}^2 \setminus \{0\} \hookrightarrow \dot{Z}$ denotes the inclusion and $e = I_2 \cdot B \in Sl_2/B \cong \mathbb{P}^1$ we obtain $\mathcal{G}$ as the fibre $\mathcal{F}(e) = j^*\mathcal{F}$. In the other direction we have $\mathcal{F} = Sl_2 \times^B \mathcal{G}$. The invariant global sections of corresponding sheaves coincide:

$$H^0(\dot{Z}, \mathcal{H}om_{\mathcal{O}_{\dot{Z}}}(\mathcal{F}, \mathcal{O}_{\dot{Z}}))^{Sl_2} = H^0(\mathbb{C}^2 \setminus \{0\}, \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}}(\mathcal{G}, \mathcal{O}_{\mathbb{C}^2\setminus\{0\}}))^B.$$

So we take $\mathcal{F} = \mathcal{I}/\mathcal{I}^2$ and are interested in determining the dual of $j^*\mathcal{F}$:
As $\mathcal{O}_{\mathbb{C}^2} = \mathbb{C}[\lambda, \mu]$ and $\mathbb{C}^2 \setminus \{0\} = \mathbb{C}^2 \setminus \{0\}_\lambda \cup \mathbb{C}^2 \setminus \{0\}_\mu$ the structure sheaf is given by

$$\begin{aligned}
\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}(\mathbb{C}^2 \setminus \{0\}_\lambda) &= \mathbb{C}[\lambda, \lambda^{-1}, \mu], \\
\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}(\mathbb{C}^2 \setminus \{0\}_\mu) &= \mathbb{C}[\lambda, \mu, \mu^{-1}].
\end{aligned}$$

In our case the inclusion is $j\colon \mathbb{C}^2 \setminus \{0\} \to \dot{Z},\ (\lambda, \mu) \mapsto \left(\begin{smallmatrix} \lambda & \mu \\ 0 & 0 \end{smallmatrix}\right)$, so on the level of rings we have $a \mapsto \lambda$, $b \mapsto \mu$, $c \mapsto 0$ and $d \mapsto 0$. This means, that $j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)$ is given by

$$\begin{aligned}
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_\lambda) &= \mathbb{C}[\lambda, \lambda^{-1}, \mu] < x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z >, \\
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_\mu) &= \mathbb{C}[\lambda, \mu, \mu^{-1}] < x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z >, \\
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)(\mathbb{C}^2 \setminus \{0\}_{\lambda\mu}) &= \mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}] < x_{12}, x_{13}, x_{16}, x_{22}, x_{23}, x_{26}, z > \\
&= \mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}] < x_{11}, x_{13}, x_{16}, x_{21}, x_{23}, x_{26}, z >
\end{aligned}$$

with base change $x_{11} = -\frac{\mu}{\lambda}x_{12}$ and $x_{21} = -\frac{\mu}{\lambda}x_{22}$.
To compute the dual $j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee = \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\mathbb{C}^2\setminus\{0\}})$, denote by (y_{ij}, w) the basis dual to (x_{ij}, z), i.e. $y_{ij}(x_{kl}) = \delta_{(ij)(kl)}$, $y_{ij}(z) = 0$, $w(x_{ij}) = 0$, $w(z) = 1$. Then we have

$$\begin{aligned}
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_\lambda) &= \mathbb{C}[\lambda, \lambda^{-1}, \mu] < y_{12}, y_{13}, y_{16}, y_{22}, y_{23}, y_{26}, w >, \\
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_\mu) &= \mathbb{C}[\lambda, \mu, \mu^{-1}] < y_{11}, y_{13}, y_{16}, y_{21}, y_{23}, y_{26}, w >, \\
j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee(\mathbb{C}^2 \setminus \{0\}_{\lambda\mu}) &= \mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}] < y_{12}, y_{13}, y_{16}, y_{22}, y_{23}, y_{26}, w > \\
&= \mathbb{C}[\lambda, \lambda^{-1}, \mu, \mu^{-1}] < y_{11}, y_{13}, y_{16}, y_{21}, y_{23}, y_{26}, w > .
\end{aligned}$$

with base change $y_{11} = -\frac{\lambda}{\mu}y_{12}$ and $y_{21} = -\frac{\lambda}{\mu}y_{22}$.

Computation of the global sections
The global sections $H^0(\mathbb{C}^2 \setminus \{0\}, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee)$ are the kernel of the map

$$\begin{aligned}
\varphi\colon H^0(\mathbb{C}^2 \setminus \{0\}_\lambda, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee) \oplus H^0(\mathbb{C}^2 \setminus \{0\}_\mu, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee) &\to H^0(\mathbb{C}^2 \setminus \{0\}_{\lambda\mu}, j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2)^\vee), \\
(p, q) &\mapsto p|_{\mathbb{C}^2\setminus\{0\}_{\lambda\mu}} - q|_{\mathbb{C}^2\setminus\{0\}_{\lambda\mu}}.
\end{aligned}$$

Let $\quad p = p_1y_{12} + p_2y_{13} + p_3y_{16} + p_4y_{22} + p_5y_{23} + p_6y_{26} + p_7w, \quad p_i \in \mathbb{C}[\lambda, \lambda^{-1}, \mu],$
$\quad q = q_1y_{11} + q_2y_{13} + q_3y_{16} + q_4y_{21} + q_5y_{23} + q_6y_{26} + q_7w, \quad q_i \in \mathbb{C}[\lambda, \mu, \mu^{-1}].$

Denote $p_i = \frac{p_i^N}{p_i^D}$ and $q_i = \frac{q_i^N}{q_i^D}$ with $p_i^N, q_i^N \in \mathbb{C}[\lambda,\mu]$, $p_i^D \in \mathbb{C}[\lambda]$ and $q_i^D \in \mathbb{C}[\mu]$, p_i^N, p_i^D relatively prime, as well as q_i^N, q_i^D. In $\mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$ we have

$$q = -\frac{\lambda}{\mu} q_1 y_{12} + q_2 y_{13} + q_3 y_{16} - \frac{\lambda}{\mu} q_4 y_{22} + q_5 y_{23} + q_6 y_{26} + q_7 w.$$

Thus if $i \in \{2,3,5,6,7\}$, for p and q to be equal in $\mathbb{C}[\lambda,\lambda^{-1},\mu,\mu^{-1}]$ we must have $p_i = q_i$, i.e. $p_i^N \cdot q_i^D = p_i^D \cdot q_i^N$. As p_i^N and p_i^D have no common factor, p_i^D must divide q_i^D. But p_i^D is a polynomial in λ while q_i^D is a polynomial in μ. This forces p_i^D to be constant, w.l.o.g. $p_i^D = 1$. This immediately implies $q_i^D = 1$ since q_i^N and q_i^D are coprime. We obtain $p_i^N = p_i = q_i = q_i^N \in \mathbb{C}[\lambda,\mu]$. If $i = 1$ or 4, we see $p_i = -\frac{\lambda}{\mu} q_i$, thus $\mu p_i = -\lambda q_i$. Thus $p_i^N = \lambda \tilde{p}_i^N$, $q_i^N = -\mu \tilde{p}_i^N$ and $p_i^D = 1 = q_i^D$ as before. This yields

$$\begin{aligned}\ker\varphi &= \{(\lambda p_1 y_{12} + p_2 y_{13} + p_3 y_{16} + \lambda p_4 y_{22} + p_5 y_{23} + p_6 y_{26} + p_7 w,\\ &\quad -\mu p_1 y_{11} + p_2 y_{13} + p_3 y_{16} - \mu p_4 y_{21} + p_5 y_{23} + p_6 y_{26} + p_7 w) \mid p_i \in \mathbb{C}[\lambda,\mu]\}\\ &= \mathbb{C}[\lambda,\mu] < \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w > .\end{aligned}$$

Thus $H^0(\mathbb{C}^2 \setminus \{0\}, \mathcal{H}om_{\mathcal{O}_{\mathbb{C}^2\setminus\{0\}}}(j^*(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2), \mathcal{O}_{\mathbb{C}^2\setminus\{0\}})) = \mathbb{C}[\lambda,\mu] < \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w >$ is a free module of rank 7.

Computation of invariants

Let us now consider the actions of Sl_2 and B on these modules. Let $g = \left(\begin{smallmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{smallmatrix}\right)$. Then we have $g \cdot \left(\begin{smallmatrix} x_{1i} \\ x_{2i}\end{smallmatrix}\right) = \left(\begin{smallmatrix} g_{11}x_{1i}+g_{12}x_{21} \\ g_{21}x_{1i}+g_{22}x_{2i}\end{smallmatrix}\right)$, thus

$$\begin{aligned} g \cdot x_{1i} &= g_{11}x_{1i} + g_{12}x_{2i}, & g \cdot x_{2i} &= g_{21}x_{1i} + g_{22}x_{2i},\\ g \cdot a &= g_{11}a + g_{12}c, & g \cdot c &= g_{21}a + g_{22}c,\\ g \cdot b &= g_{11}b + g_{12}d, & g \cdot d &= g_{21}b + g_{22}d\end{aligned}$$

$$\begin{aligned} g \cdot z &= g(x_{14}x_{25} - x_{15}x_{24})\\ &= (g_{11}x_{14} + g_{12}x_{24})(g_{21}x_{15} + g_{22}x_{25}) - (g_{11}x_{15} + g_{12}x_{25})(g_{21}x_{14} + g_{22}x_{24})\\ &= (g_{11}g_{22} - g_{12}g_{21})(x_{14}x_{25} - x_{15}x_{24}) = z.\end{aligned}$$

The action on the dual is determined by

$$\left.\begin{aligned} g\cdot y_{1i}(x_{1i}) &= y_{1i}(g^{-1}x_{1i}) = y_{1i}(g_{22}x_{1i} - g_{12}x_{2i}) = g_{22}\\ g\cdot y_{1i}(x_{2i}) &= y_{1i}(g^{-1}x_{2i}) = y_{1i}(-g_{21}x_{1i} + g_{11}x_{2i}) = -g_{21}\end{aligned}\right\} \quad\Rightarrow\quad g\cdot y_{1i} = g_{22}y_{1i} - g_{21}y_{2i},$$

$$\left.\begin{aligned} g\cdot y_{2i}(x_{1i}) &= y_{2i}(g^{-1}x_{1i}) = y_{2i}(g_{22}x_{1i} - g_{12}x_{2i}) = -g_{12}\\ g\cdot y_{2i}(x_{2i}) &= y_{2i}(g^{-1}x_{2i}) = y_{2i}(-g_{21}x_{1i} + g_{11}x_{2i}) = g_{11}\end{aligned}\right\} \quad\Rightarrow\quad g\cdot y_{2i} = -g_{12}y_{1i} + g_{11}y_{2i},$$

$$g\cdot w(z) = w(g^{-1}z) = w(z) \quad\Rightarrow\quad g\cdot w = w.$$

Correspondingly, over $\mathbb{C}^2 \setminus \{0\}$, the action of $g = \left(\begin{smallmatrix} t & u \\ 0 & t^{-1}\end{smallmatrix}\right)$ is

$$\begin{aligned} g\cdot\lambda &= t\lambda, & g\cdot x_{1i} &= tx_{1i} + ux_{2i}, & g\cdot y_{1i} &= t^{-1}y_{1i},\\ g\cdot\mu &= t\mu, & g\cdot x_{2i} &= t^{-1}x_{2i}, & g\cdot y_{2i} &= -uy_{1i} + ty_{2i},\\ & & g\cdot z &= z, & g\cdot w &= w.\end{aligned}$$

Now we have $B = TU$ with torus $T = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)\right\}$ and unipotent radical $U = \{\left(\begin{smallmatrix} 1 & u \\ 0 & 1\end{smallmatrix}\right)\}$. Thus we can compute the B-invariants stepwise:

$$\mathbb{C}[\lambda,\mu] < \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w >^B = (\mathbb{C}[\lambda,\mu] < \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w >^U)^T.$$

Let $\underline{u} = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$:

$$\left.\begin{array}{ll} \underline{u} \cdot \lambda = \lambda & \underline{u} \cdot \lambda y_{12} = \lambda y_{12} \\ \underline{u} \cdot \mu = \mu & \underline{u} \cdot y_{13} = y_{13} \\ \underline{u} \cdot w = w & \underline{u} \cdot y_{16} = y_{16} \end{array}\right\} \text{invariants}$$

$$\left.\begin{array}{l} \underline{u} \cdot \lambda y_{22} = \lambda(-uy_{12} + y_{22}) = -u\lambda y_{12} + \lambda y_{22} \\ \underline{u} \cdot y_{23} = -uy_{13} + y_{23} \\ \underline{u} \cdot y_{26} = -uy_{16} + y_{26} \end{array}\right\} \text{cannot be combined to form invariants.}$$

So we have $\quad \mathbb{C}[\lambda, \mu] < \lambda y_{12}, y_{13}, y_{16}, \lambda y_{22}, y_{23}, y_{26}, w >^U = \mathbb{C}[\lambda, \mu] < \lambda y_{12}, y_{13}, y_{16}, w > .$

To compute the T-invariants, let $\underline{t} = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$. We obtain

$$\begin{array}{lll} \text{degree } 1: & \text{invariants:} & \text{degree } -1: \\ \underline{t} \cdot \lambda = t\lambda & \underline{t} \cdot w = w & \underline{t} \cdot y_{13} = t^{-1} y_{13} \\ \underline{t} \cdot \mu = t\mu & \underline{t} \cdot \lambda y_{12} = t\lambda t^{-1} y_{12} = \lambda y_{12} & \underline{t} \cdot y_{16} = t^{-1} y_{16} \end{array}$$

This yields the invariants w, λy_{12}, λy_{13}, μy_{13}, λy_{16} and μy_{16}. So we have computed

$$\begin{aligned} H^0(\dot{Z}, \mathrm{Hom}(\dot{\mathcal{I}}/\dot{\mathcal{I}}^2, \mathcal{O}_{\dot{Z}}))^{Sl_2} &= H^0(\mathbb{C}^2 \setminus \{0\}, \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}))^B \\ &= \mathbb{C} < \lambda y_{12}, \lambda y_{13}, \mu y_{13}, \lambda y_{16}, \mu y_{16}, w > . \end{aligned}$$

This means that $T_Z Sl_2$-Hilb$(\mu^{-1}(0))$ is 6-dimensional and therefore the orbit component of the invariant Hilbert scheme is a smooth connected component. □

5.2. **Connectivity.** In general, the invariant Hilbert scheme can be disconnected. To examine connectivity we look at $\mathbb{C}^*$-actions:
If there is a $\mathbb{C}^*$-action on X which commutes with the G-action, it descends to a $\mathbb{C}^*$-action on $X/\!\!/G$ so that the quotient map $X \to X/\!\!/G$ is $\mathbb{C}^*$-equivariant. In this case, one way to find out if the invariant Hilbert scheme is connected is to compute the induced $\mathbb{C}^*$-action on $\mathrm{Hilb}_h^G(X)$ and to determine all fixed points of $\mathbb{C}^* \circlearrowright X/\!\!/G$. The Hilbert-Chow morphism is proper and $\mathbb{C}^*$-equivariant, therefore for every fixed point in the image there is at least one fixed point in every connected component of the fibre of its preimage.

Remark. Let $(X/\!\!/G)_*$ denote the flat locus of the quotient map. Since $\eta|_{\eta^{-1}((X/\!\!/G)_*)}$ is an isomorphism, every irreducible component of the invariant Hilbert scheme different from $\mathrm{Hilb}_h^G(X)^{orb} = \overline{\eta^{-1}((X/\!\!/G)_*)}$ only contains points of the fibres over $X/\!\!/G \setminus (X/\!\!/G)_*$. If one can show that all connected components of these fibres meet the orbit component, and additionally one knows the orbit component to be smooth, then there cannot be any further component. In this case $\mathrm{Hilb}_h^G(X) = \mathrm{Hilb}_h^G(X)^{orb}$ is connected.

Connectedness of Sl_2-Hilb$(\mu^{-1}(0))$.

Proposition 5.4. *The invariant Hilbert scheme Sl_2-Hilb$(\mu^{-1}(0))$ is connected, hence it coincides with its orbit component and we have*

$$Sl_2\text{-Hilb}(\mu^{-1}(0)) = Sl_2\text{-Hilb}(\mu^{-1}(0))^{orb} = \{(A, U) \in \mathcal{O}_{[2^2,1^2]} \times \mathrm{Grass}_{iso}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset U\}.$$

Proof We consider the action of $\mathbb{C}^*$ on $\mu^{-1}(0)$ by scalar multiplication and the induced action on $\mu^{-1}(0)/\!\!/Sl_2 = \overline{\mathcal{O}}_{[2^2,1^2]}$. For $t \in \mathbb{C}$ and $M \in \mu^{-1}(0)$ we have $(tM)^t J(tM)Q = t^2(M^t JMQ)$, thus the action on the quotient is multiplication with t^2. Further $A \in \overline{\mathcal{O}}_{[2^2,1^2]}$ is invariant if and only if $A = 0$, so all fixed points of Sl_2-Hilb$(\mu^{-1}(0))$ map to 0.

The induced action on Sl_2-Hilb$(\mu^{-1}(0))$ maps Z to tZ. If Z is an Sl_2-invariant subscheme of $\mu^{-1}(0)$, then tZ is also Sl_2-invariant because the action of Sl_2 commutes with scalar multiplication. Secondly, the global sections of Z and tZ and their isotypic decompositions coincide, so indeed $tZ \in Sl_2$-Hilb$(\mu^{-1}(0))$.
The following lemma shows that set of $\mathbb{C}^*$-fixed points in Sl_2-Hilb$(\mu^{-1}(0))$ is $\mathrm{Grass}_{iso}(2, \mathbb{C}^6)$, the fibre of Sl_2-Hilb$(\mu^{-1}(0))^{orb}$ over zero. Consequently, $\eta^{-1}(0)$ has no further components, and the same is true for Sl_2-Hilb$(\mu^{-1}(0))$. This shows proposition 5.4 and concludes the proof of theorem 1.1. □

Lemma 5.5. *The set of fixed points in* Sl_2-Hilb$((\mathbb{C}^2)^{\oplus 6})$ *under the* $\mathbb{C}^*$*-action is isomorphic to the Grassmannian* $\mathrm{Grass}(2, \mathbb{C}^6)$. *The* $\mathbb{C}^*$*-fixed points in* Sl_2-Hilb$(\mu^{-1}(0))$ *correspond to the points of* $\mathrm{Grass}_{iso}(2, \mathbb{C}^6)$.

Proof Let $Z \subset (\mathbb{C}^2)^{\oplus 6}$ be a $\mathbb{C}^*$-fixed point in Sl_2-Hilb$((\mathbb{C}^2)^{\oplus 6})$ or Sl_2-Hilb$(\mu^{-1}(0))$, equivalently its corresponding ideal $\mathcal{I}$ is homogeneous. Since then the Hilbert-Chow morphism maps Z to 0, all 2×2-minors of each element in Z vanish. Hence $\mathcal{I}$ contains all the 15 minors Λ^{ij}.
Now let us analyse the homogeneous invariant ideals $\mathcal{I}$ in $R = \mathbb{C}[x_{11}, \ldots, x_{26}]$, containing all Λ^{ij}, with isotypic decomposition $R/\mathcal{I} = \bigoplus_{d \in \mathbb{N}_0} (d+1)V_d$, where $V_d = \mathbb{C}[x, y]_d$ denotes the representation of Sl_2 of dimension $d + 1$. Afterwards we will restrict to ideals containing XQX^t, which correspond to ideals in $R/(XQX^t)$, and which are the fixed points of Sl_2-Hilb$(\mu^{-1}(0))$.
The representation $(\mathbb{C}^2)^{\oplus 6} = \mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)$ consists of 6 copies of V_1, so $R = \bigoplus_{n \in \mathbb{N}_0} S^n(6V_1)$. Since $R = \bigoplus_{n \in \mathbb{N}_0} S^n(\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)^*)$ is graded and $\mathcal{I}$ is homogeneous, $R/\mathcal{I}$ is still a graded object, so that we can analyse it by degree. The invariance of $\mathcal{I}$ guarantees that $\mathcal{I}_1$ is a subrepresentation of $\mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)^*$, i.e. there is a sub-vectorspace $V \subset \mathbb{C}^6$ such that $\mathcal{I}_1 = \mathrm{Hom}(V, \mathbb{C}^2)^*$. The isotypic decompostion of $R/\mathcal{I}$ requires exactly two copies of V_1, and they must already come from $R_1/\mathcal{I}_1$, since no such copy can be contribued or killed by generators of higher degree. If the dimension of V were 5 or 6 then $R_1/\mathcal{I}_1$ would be too small and if $\dim V \leq 3$ then $R_1/\mathcal{I}_1$ would be too big. Thus we know that $\dim V = 4$, so that after a coordinate transformation we can write $\mathcal{I} \supset \mathcal{J} = (x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6, x_1y_2 - y_1x_2)$, since the other 2×2-minors $x_iy_j - y_jx_i$ do not contribute to the generation of the ideal. Then $R/\mathcal{J} \cong \mathbb{C}[x_1, y_1, x_2, y_2]/(x_1y_2 - y_1x_2)$ is the coordinate ring of a flat deformation of Sl_2 an has isotypic decomposition $\bigoplus_{n \in \mathbb{N}_0} (n+1)V_n$ as desired. Hence we need no further generators and $\mathcal{I} = \mathcal{J}$.
So the fixed points in Sl_2-Hilb$((\mathbb{C}^2)^{\oplus 6})$ under the $\mathbb{C}^*$-action correspond to the choice of a 4-dimensional subspace of $\mathbb{C}^6$, which is parameterised by the Grassmannian $\mathrm{Grass}(4, \mathbb{C}^6)$, which coincides with $\mathrm{Grass}(\mathbb{C}^6, 2)$ and $\mathrm{Grass}(2, \mathbb{C}^6)$.
For Z to be contained in $\mu^{-1}(0)$ we have to pick only those ideals which contain XQX^t, so that we have $MQM^t = 0$ for every $M \in Z$. We interpret $M \in (\mathbb{C}^2)^{\oplus 6}$ as a map $\mathbb{C}^6 \to \mathbb{C}^2$. The fact $M \in Z = \mathrm{Spec}(R/\mathcal{I})$ means that M vanishes on V, so we can interpret it as a map $\mathbb{C}^6/V \to \mathbb{C}^2$. As the inner product on $(\mathbb{C}^2)^{\oplus 6}$ is induced by the inner product on $\mathbb{C}^6$, the condition $MQM^t = 0$ for every $M \in Z$ is equivalent to the vanishing of v^tQv for all $v \in \mathbb{C}^6/V$. This show that $\mathcal{I} \supset (XQX^t)$ if and only if $\mathbb{C}^6/V$ is an isotropic subspace of $\mathbb{C}^6$.

□

Remark. Sl_2-Hilb$((\mu^{-1}(0))$ is a subscheme of the Hilbert scheme Sl_2-Hilb$((\mathbb{C}^2)^{\oplus 6})$. The calculation of the fixed points suggests that Sl_2-Hilb$((\mathbb{C}^2)^{\oplus 6})$ contains the whole Grassmannian as the fibre over 0. Indeed, Sl_2-Hilb$((\mathbb{C}^2)^6) = \{((\mathbb{C})^2)^{\oplus 6} \times \mathrm{Grass}(2, \mathbb{C}^6) \mid \mathrm{im}\, A^t \subset W\}$ as forthcoming work by Terpereau will show.

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